# DATA ANALYSIS

Week 3: Variability

# logistics: quiz 2 / problem set #1

- quiz 2
  - bar graph / histogram question was regraded
- problem set #1
  - going forward, please submit a PDF of your document with link to sheet as before

# logistics: midterm 1

- Feb 23rd 2024 (Friday)
- content: Week 1 through Week 5
- two parts
  - in-person conceptual portion (quiz-like)
    - closed book (+ help sheet)
  - take-home computational portion (problem set-like: due Monday)
    - open book but NOT open person
- practice questions will be made available before Week 5

# today's agenda

variability

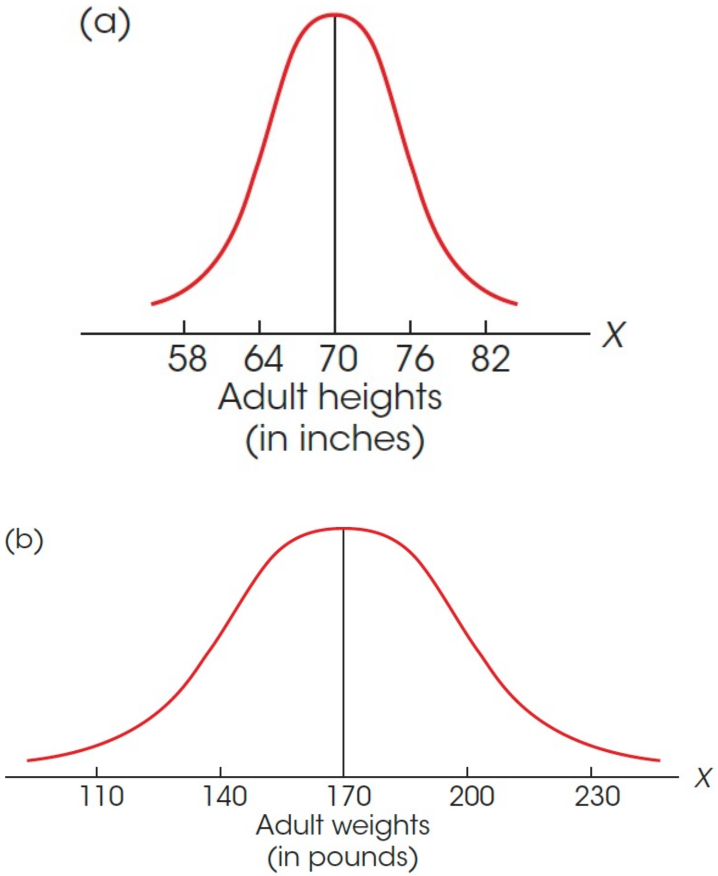
z-scores

# recap of fitting models

- models are fit to data: data = model + error
- we fit “central tendencies”/models to the data (mean / median / mode)
- we calculated “errors”/distances between the data and our model(s)
  - sum of squared errors (SSE or SS): $\sum_{i=1}^{N}(X_i - \mu)^2$
  - mean of squared errors (MSE): $\frac{\sum_{i=1}^{N}(X_i - \mu)^2}{N} = \frac{SS}{N}$
  - root mean squared error (RMSE): $\sqrt[2]{\frac{\sum_{i=1}^{N}(X_i - \mu)^2}{N}} = \sqrt{MSE}$

# variability

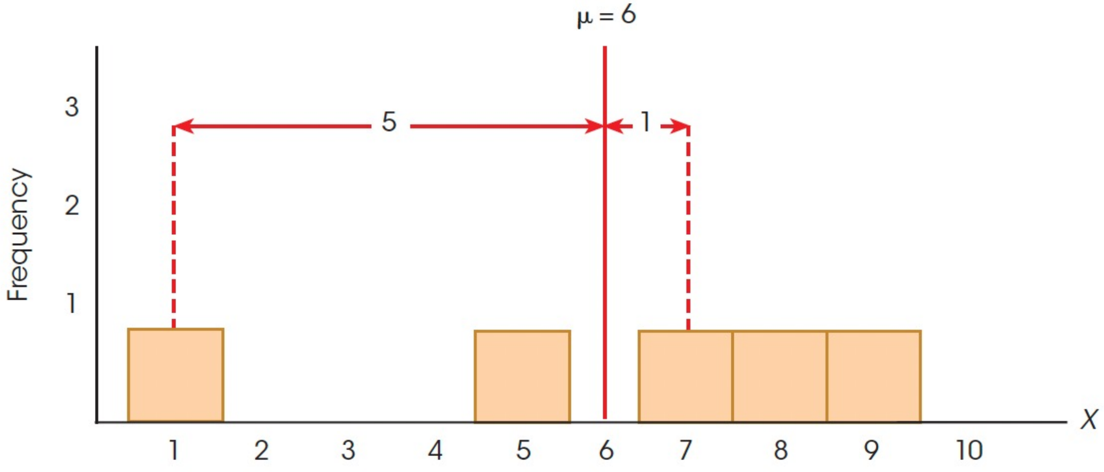
- variability describes the spread of scores in a distribution
- measures of variability
  - range = maximum – minimum
  - variance = mean squared error from the mean (MSE or $\sigma^2$) = average of ***squared*** <u>distances</u>/errors from the mean
  - standard deviation = root mean squared error from the mean (RMSE or $\sigma$) = average <u>distance</u>/error from the mean **in original units**
- variance and standard deviation are defined relative to the mean, i.e., how well does the mean fit the data?



# visual inspection

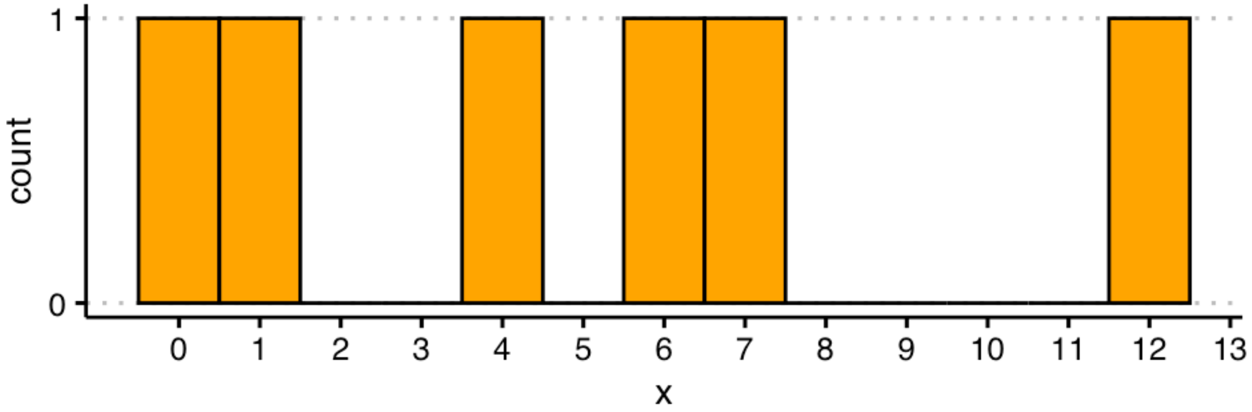
- we can estimate/calculate the mean
- the farthest score is 5 points away
- the closest score is 1 point away
- on average, scores are likely $\frac{5+1}{2}$ away = 3 points away
- what is our actual estimate of standard deviation for these scores?

$$\sqrt[2]{\frac{\sum_{i=1}^{N}(X_i-\mu)^2}{N}} = \sqrt{\frac{25+1+1+4+9}{5}} = 2.83$$



# activity

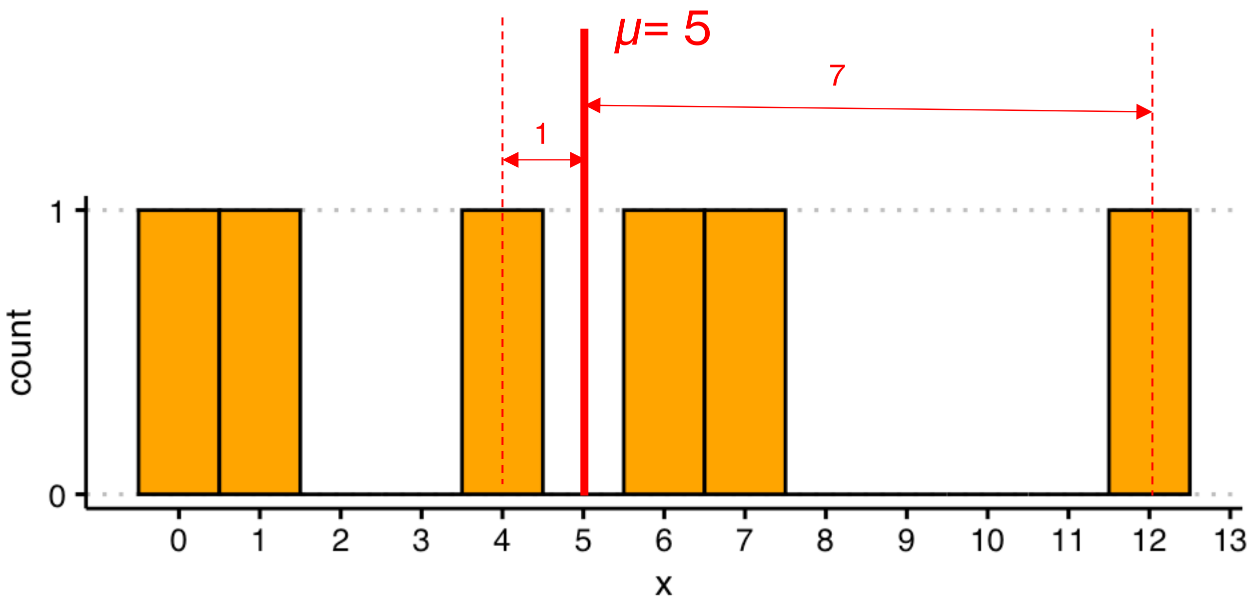
- 6 scores: 12, 0, 1, 7, 4, and 6
- calculate the mean
- visually estimate the standard deviation



# activity

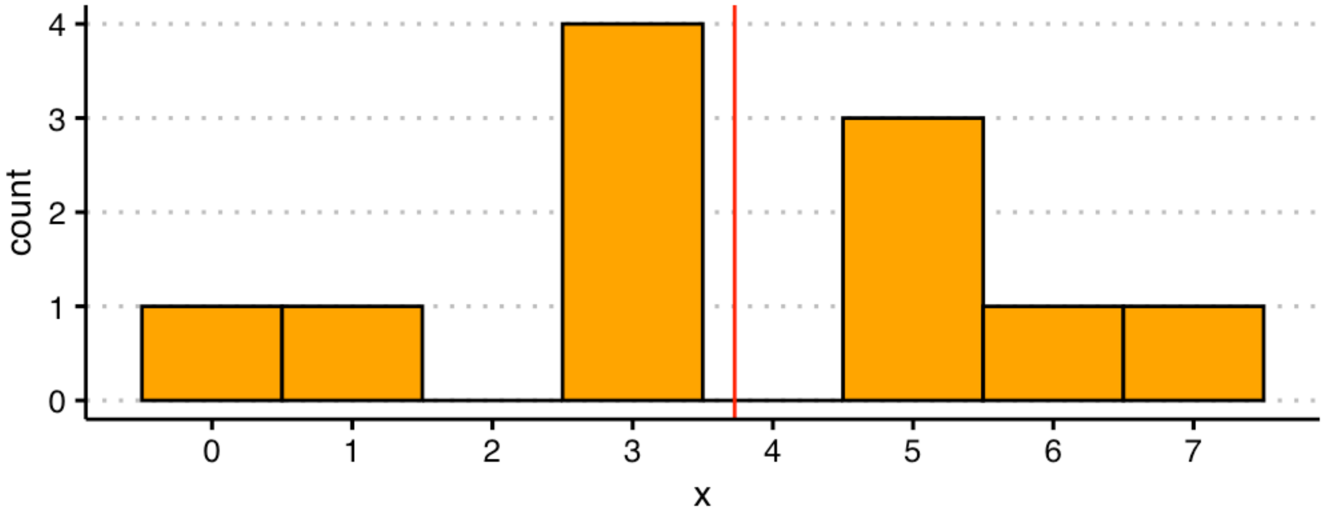
- mean = 5 points
- farthest score is 7 points away
- closest score is 1 point away
- sd estimate = 7 + 1 / 2 = 4
- actual sd = $\sqrt[2]{\frac{\sum_{i=1}^{N}(X_i-\mu)^2}{N}} = \sqrt{\frac{49+25+16+4+1+1}{6}}$

= $\sqrt{\frac{96}{6}} = 4$



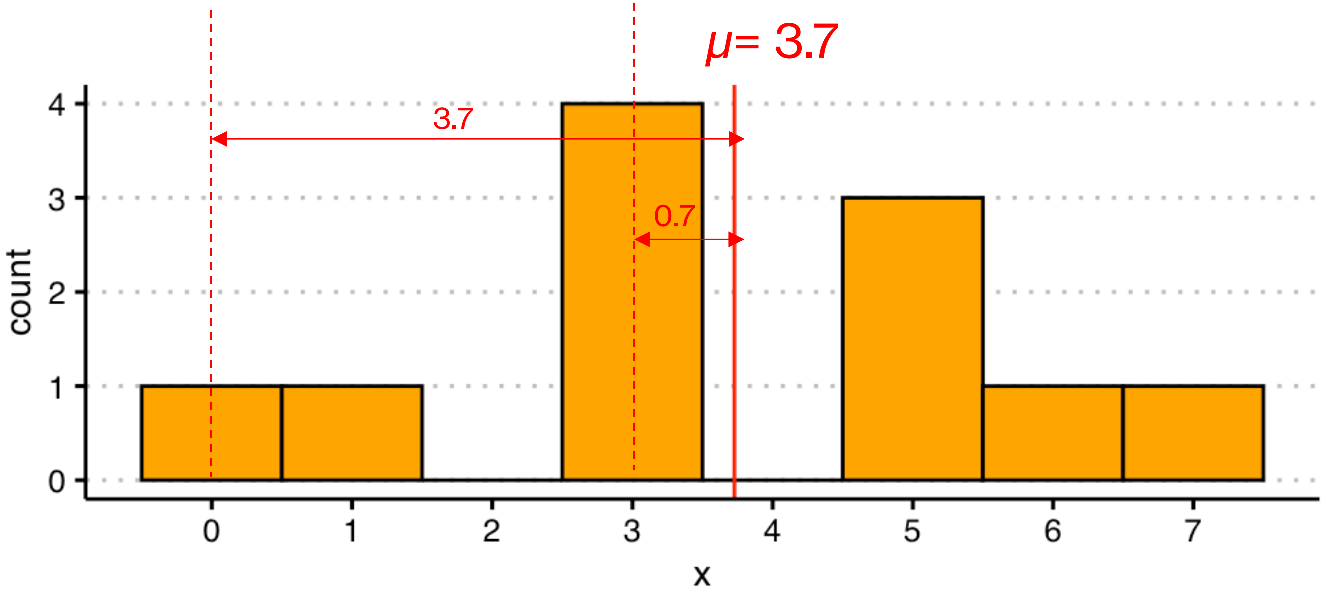
# activity

- 5,5,5, 3,3,3,3,6, 7, 1, 0
- calculate the mean
- visually estimate the standard deviation



# activity

- mean is 3.7
- farthest score is 3.7 points away
- closest score is 0.7 point away
- sd estimate = 3.7 + 0.7 / 2 = 2.2
- but more scores are on the closer side so slightly less than 2.2 is likely
- actual sd = $\sqrt[2]{\frac{\sum_{i=1}^{N}(X_i-\mu)^2}{N}} = 2.00$



# SSE: definitional vs. computational formulas

$$\sum(X-\mu)^2 = \sum X^2 - \frac{(\sum X)^2}{N}$$

definitional formula

computational formula

$$\sum(X-\mu)^2 = \sum(X^2+\mu^2-2X\mu) = \sum X^2 + \sum \mu^2 - 2\sum X\mu = \sum X^2 + N\mu^2 - 2\mu\sum X =$$

$$= \sum X^2 + N\mu^2 - 2\,\frac{\sum X}{N}\sum X = \sum X^2 + N\mu^2 - 2\,\frac{(\sum X)^2}{N} = \sum X^2 + N\frac{\sum X}{N}\frac{\sum X}{N} - 2\,\frac{(\sum X)^2}{N} = \sum X^2 + \frac{(\sum X)^2}{N} - 2\,\frac{(\sum X)^2}{N}$$

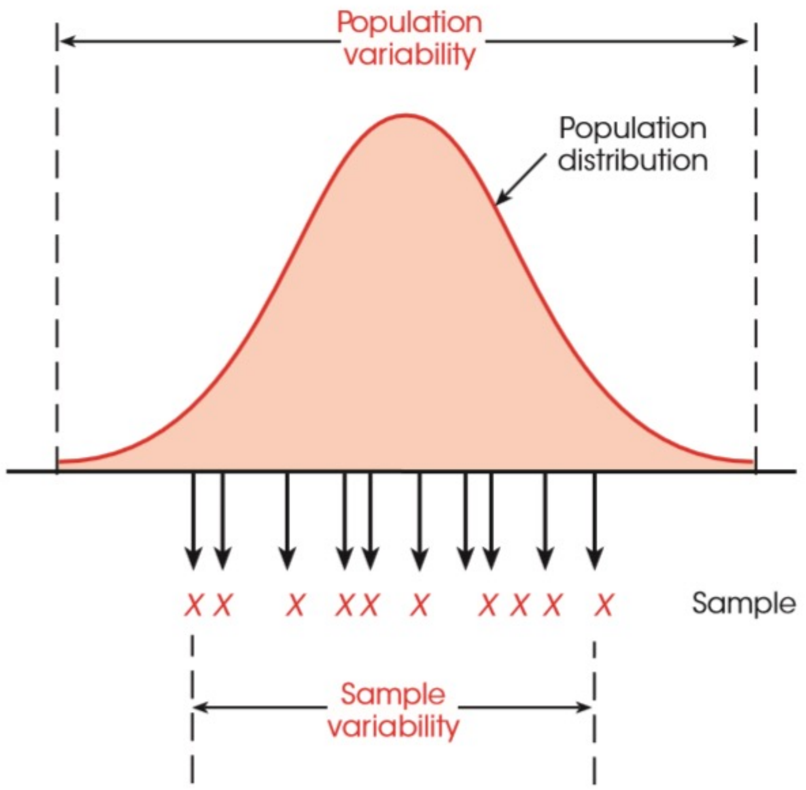
$$= \sum X^2 - \frac{(\sum X)^2}{N}$$

only for your curiosity, stick to definitional formula for this class: easier to remember and understand

# questions?

# from populations to samples

- we have been talking about central tendencies and spread for populations, but we hardly ever have access to the populations!
- sample means (*M*) contribute to sample-based estimates of variance ($s^2$) and standard deviation ($s$)
- sampling tends to focus more on “typical” scores, so we tend to miss out on extreme scores from the population
- as a result, samples tend to underestimate population variability



# a demonstration: small population

- consider an island population (N = 6) where people were asked to report how many trees they own on the island
- 2 people owned no trees, 2 people owned 3 trees each, and 2 people owned 9 trees each!
- we calculate the mean and standard deviation of trees owned for this population

B2 =A2-$A$10

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | X | data-mu | squared errors | MSE (variance) | RMSE (sd) |
| 2 | 0 | -4 | 16 | 14 | 3.741657387 |
| 3 | 0 | -4 | 16 | | |
| 4 | 3 | -1 | 1 | | |
| 5 | 3 | -1 | 1 | | |
| 6 | 9 | 5 | 25 | | |
| 7 | 9 | 5 | 25 | | |
| 8 | | | | | |
| 9 | Mu | | SSE | | |
| 10 | 4 | | 84 | | |

# a demonstration: small samples

- now we take all possible samples of size 2 from this population
- calculate the mean M for each sample
- average M from all possible samples is equal to the population M: mean is an unbiased statistic!

| sample number | X1 | X2 | M |
|---|---|---|---|
| 1 | 0 | 0 | 0 |
| 2 | 0 | 3 | 1.5 |
| 3 | 0 | 9 | 4.5 |
| 4 | 3 | 0 | 1.5 |
| 5 | 3 | 3 | 3 |
| 6 | 3 | 9 | 6 |
| 7 | 9 | 0 | 4.5 |
| 8 | 9 | 3 | 6 |
| 9 | 9 | 9 | 9 |
| | | | |
| | | | M_avg |
| | | | 4 |

B2 fx =

| | A |
|---|---|
| 1 | X |
| 2 | 0 |
| 3 | 0 |
| 4 | 3 |
| 5 | 3 |
| 6 | 9 |
| 7 | 9 |
| 8 | |
| 9 | Mu |
| 10 | 4 |

# a demonstration: small samples

- calculate the variance (MSE) of each sample

$$= \frac{\sum_{i=1}^{N}(X_i - M_{sample})^2}{n}$$

- average variance is LOWER than the population variance: variance is a biased statistic!

| sample number | X1 | X2 | M | variance_biased |
|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 |
| 2 | 0 | 3 | 1.5 | 2.25 |
| 3 | 0 | 9 | 4.5 | 20.25 |
| 4 | 3 | 0 | 1.5 | 2.25 |
| 5 | 3 | 3 | 3 | 0 |
| 6 | 3 | 9 | 6 | 9 |
| 7 | 9 | 0 | 4.5 | 20.25 |
| 8 | 9 | 3 | 6 | 9 |
| 9 | 9 | 9 | 9 | 0 |
| | | | | |
| | | | M_avg | var_biased_avg |
| | | | 4 | 7 |

| MSE (variance) | RMSE (sd) |
|---|---|
| 14 | 3.741657387 |

# a demonstration: small samples

- we need to penalize the sample variance so that it accurately estimates the population variance
- we need to make variance (MSE) a larger number

$$\frac{\sum_{i=1}^{N}(X_i - M_{sample})^2}{n}$$

- we can decrease the the denominator: divide by (n – 1) instead

$$s^2 = \frac{\sum_{i=1}^{N}(X_i - M_{sample})^2}{n-1}$$

- also called the Bessel's correction

| sample number | X1 | X2 | M | variance_biased | variance_unbiased |
|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 |
| 2 | 0 | 3 | 1.5 | 2.25 | 4.5 |
| 3 | 0 | 9 | 4.5 | 20.25 | 40.5 |
| 4 | 3 | 0 | 1.5 | 2.25 | 4.5 |
| 5 | 3 | 3 | 3 | 0 | 0 |
| 6 | 3 | 9 | 6 | 9 | 18 |
| 7 | 9 | 0 | 4.5 | 20.25 | 40.5 |
| 8 | 9 | 3 | 6 | 9 | 18 |
| 9 | 9 | 9 | 9 | 0 | 0 |
| | | | | | |
| | | | M_avg | var_biased_avg | var_unbiased_avg |
| | | | 4 | 7 | 14 |

| MSE (variance) | RMSE (sd) |
|---|---|
| 14 | 3.741657387 |

# why (n-1)? degrees of freedom

- df = number of values that are *free to vary* in the calculation of a statistic
- for populations, we use the population mean ($\mu$) to compute deviation scores (X - $\mu$)
- however, for samples, $\mu$ is unknown and we estimate it using our sample mean *M*
- computing M restricts the scores that went into the calculation
  - why? because changing even a single score would change M
  - if M is known, you only need to know n-1 scores to find the last score
  - only n-1 scores are free to vary once M is known

# an example

- if the mean of quiz scores for 5 students is 9 points and four students' scores are 8, 10, 8, and 9, what is the score of the fifth student?

# populations vs. samples

**populations**

- population variance $(\sigma^2) = \frac{\sum(X-\mu)^2}{N} = \frac{SS}{N}$
- population standard deviation $(\sigma)$ = $\sqrt{\frac{\sum(X-\mu)^2}{N}} = \sqrt{\frac{SS}{N}}$

**samples**

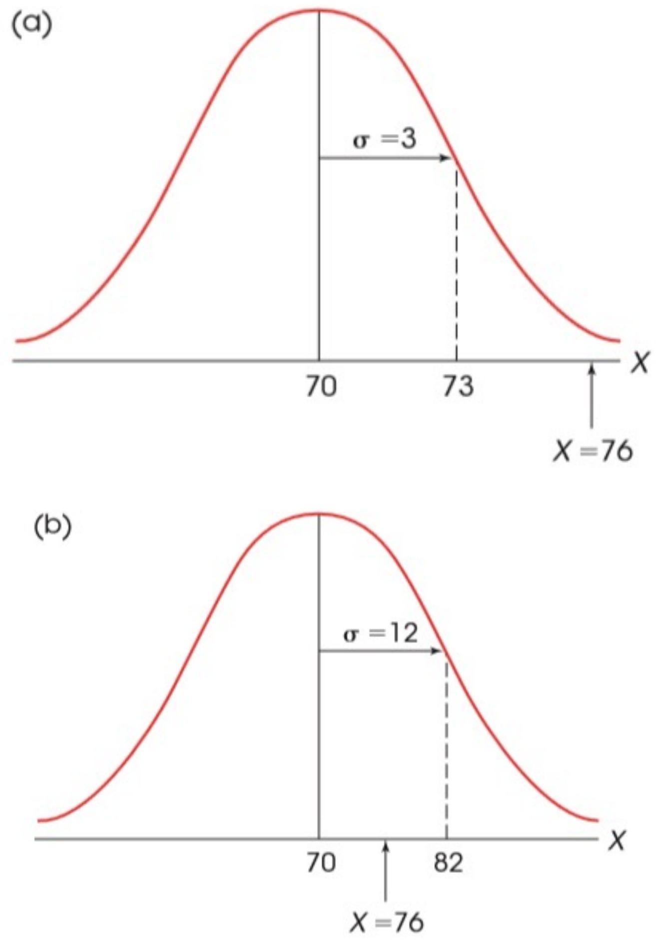
- sample variance $(s^2)$ = $\frac{\sum(X-M)^2}{n-1} = \frac{SS}{n-1} = \frac{SS}{df}$
- sample standard deviation $(s)$ = $\sqrt{\frac{\sum(X-M)^2}{n-1}} = \sqrt{\frac{SS}{n-1}} = \sqrt{\frac{SS}{df}}$

# questions?

- explore the variability sheet

# locating scores within distributions

- we have used means and standard deviations as ways to summarize distributions
- but, if you wanted to know how well you performed on a test, how would you apply this knowledge of the distribution to know how well you did?
- means and standard deviations together can be informative in describing a data point's relationship to the distribution

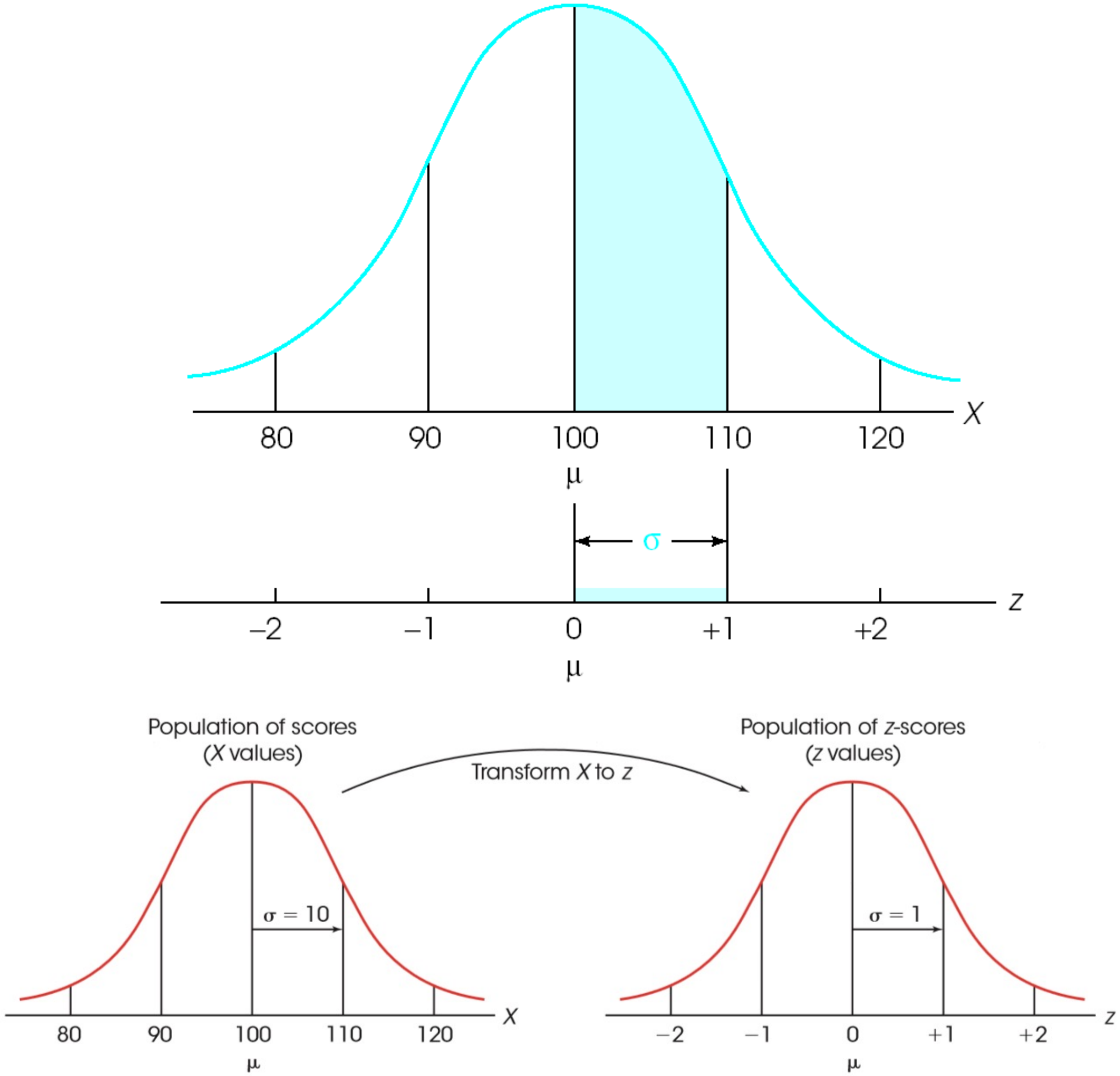


# z-scores

- z-scores are a way to understand how far away a score is from the mean, in standard deviation units

$$z = \frac{X - \mu}{\sigma}$$

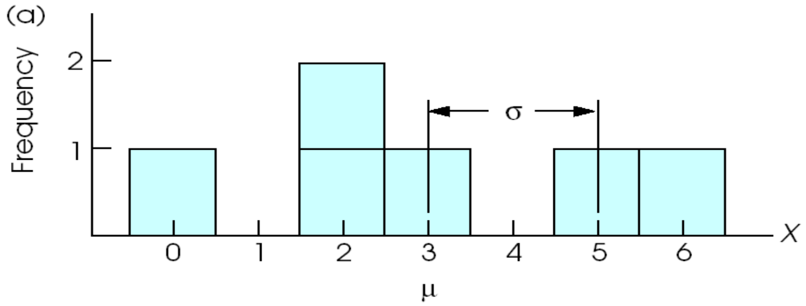
  - calculate “distances” or deviation scores and divide by the standard deviation
  - z-score is essentially a ratio that is asking: how extreme is my score relative to the average distance I can expect based on this distribution?
- any distribution can be transformed into a distribution of z-scores



# calculating z-scores

$$z = \frac{X - \mu}{\sigma}$$

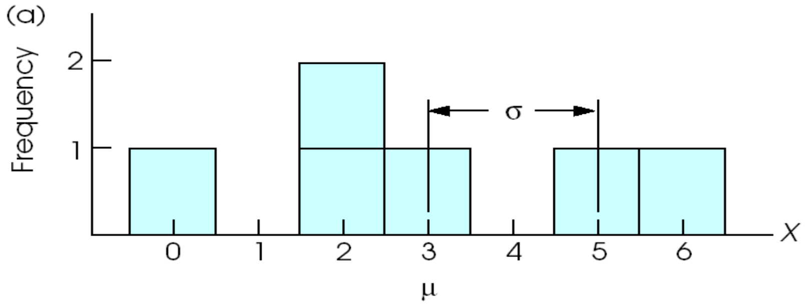
- six scores, calculate $\mu$, $\boldsymbol{\sigma}$, and z

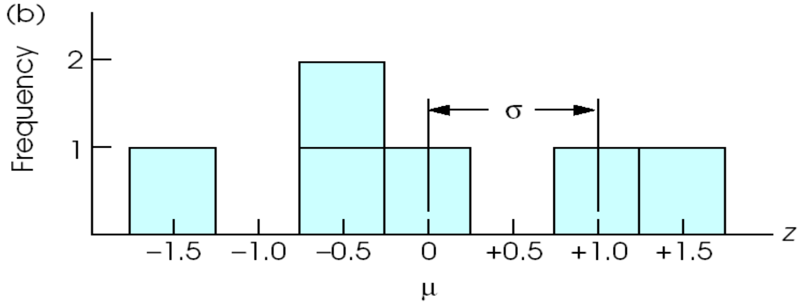


# calculating z-scores

$$z = \frac{X - \mu}{\sigma}$$

- six scores, calculate $\mu$, $\boldsymbol{\sigma}$, and z




| X | mu | X-mu | squared_errors | MSE | RMSE | z |
|---|---|---|---|---|---|---|
| 0 | 3 | -3 | 9 | 4 | 2 | -1.5 |
| 6 | | 3 | 9 | | | 1.5 |
| 5 | | 2 | 4 | | | 1 |
| 2 | | -1 | 1 | | | -0.5 |
| 3 | | 0 | 0 | | | 0 |
| 2 | | -1 | 1 | | | -0.5 |

solution sheet

# next time

- **before** class
  - *watch*: Variability and z-scores
  - *prep*: chapter 6 (specific sections – see course website)
  - *try*: problem set #2 (chapter 4 and 5 problems)
- **during** class
  - deep dive into the normal distribution